THE OPEN UNIVERSITY

TMA FORM (PT3)

STUDENT

2

SECTION 1 STUDENT TO COMPLETE

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Course and Assignment No. 12: SM 352 05

Please take particular care to enter these two numbers correctly

*THIS SHOULD NOT BE YOUR COURSE TUTOR'S NAME AND NUMBER

SECTION 2 TUTOR TO COMPLETE

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24 29

24/10

Question Scores

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Overall Score
26	30	24	12																	92
30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62	64	66	68	70

TUTOR'S COMMENTS AND ADVICE TO STUDENT

An excellent TMA. Very well done. See also my notes on your script.

Good luck in the exam!

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The Open University

Science Supplementary Material

TMA

Course and assignment number:

Tutor Marked Assignment

SM352 05

Make sure you know how to complete and send in your TMA and PT3 form: detailed instructions are given in your student handbook (or supplement).

Covering: **Unit 6 and Units 12 to 15**

Cut-off date:
Friday 5 October 1990

Question 1

This question relates to Units 6 and 12 and carries 30 per cent of the marks for this assignment.

Give an account of the similarities and differences between the fields **D** and **H** due to static charges and steady currents respectively. Your account should be about two sides of A4 paper in length, and should concentrate on the following points:

(a) the definitions of **D** and **H**;

(b) the differences between the ways in which **D** and **H** are related to their respective sources;

(c) the differences between the boundary conditions satisfied by **D** and **H** (simplest cases only);

(d) the reason why those problems in which **H** is easily calculated are not necessarily similar to those in which **D** is easily calculated (e.g. to calculate **H** for a toroidal solenoid is easy, yet to calculate **D** for a toroidal charge is not).

Where relevant, you should quote equations to illustrate the point you are discussing, but your main aim should be to show that you have grasped the physical significance of the equations. You may find it helpful to refer to Table 1, pp. 12 of Unit 12, but your account should not simply list the equations given in the Table.

Question 2

This question carries 30 per cent of the marks for this assignment, and covers Units 13/14.

(a) Starting from Maxwell's equations for free space, derive the wave equation

$$\nabla^2\mathbf{B} = \mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

[You may require the vector identity $\nabla\times(\nabla\times\mathbf{X}) = \nabla(\nabla\cdot\mathbf{X}) - \nabla^2\mathbf{X}$.]

Show that if the complex wave $B = B_0\hat{y}\exp(j(\omega t - kz))$ satisfies the wave equations, then ω and k have a constant relationship. Interpret your equation relating ω and k.

(b) Prove that there is a complex wave

$$\mathbf{E} = E_0\hat{x}\exp(j(\omega t - kz))$$

corresponding to the wave **B**.
[Hint: find curl **B**]

Hence show that in free space $E = Z_0 H$, where Z_0 is called the wave impedance of free space. Prove Z_0 has dimensions [Ω] and calculate its value.

(c) Show that Maxwell's equations in free space are also obeyed by the new fields **E**′ and **B**′, where

$$\mathbf{E}' = \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta; \quad \mathbf{B}' = -\frac{\mathbf{E}}{c}\sin\theta + \mathbf{B}\cos\theta$$

and θ is a constant.

Question 3

This question relates mainly to Unit 15 and carries 28 per cent of the marks for this assignment.

A ray of light is incident on a semicircular half cylinder made of glass of refractive index 1.5, as shown in cross-section in Figure 1.

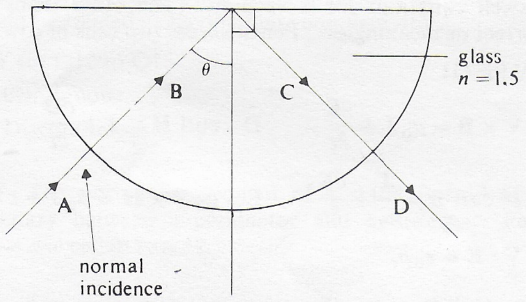

FIGURE 1

(a) What is the angle θ such that the emerging ray is fully polarized?

(b) Ignoring multiple reflections within the glass and absorption by the glass, what is the amplitude of E_σ on exit from the glass if E_σ was of unit magnitude for the incoming light ray?

(c) With the same assumptions as in (a) and (b), calculate the *energy flux* at points A, B, C and D in Figure 1. Explain why E may be greater at D than at C without contradicting the law of conservation of energy.

Question 4

This general revision question carries 12 per cent of the marks for this assignment.

In establishing the properties of the Poynting vector, the text of Units 13/14, page 25, quotes the 'readily established identity':

$$\text{div}\,(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\text{curl}\,\mathbf{A}) - \mathbf{A}\cdot(\text{curl}\,\mathbf{B})$$

Verify this equation, showing all the steps in the argument, and use it to show that if $\mathbf{A} = A\hat{\mathbf{z}}$, where A is constant, then

$$\text{div}\,(\mathbf{A}\times\mathbf{B}) = A\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right)$$

for any vector field **B**.

(In this question, marks will be deducted for non-use or misuse of consistent vector notation.)

TMA SM352 05 JOHN.A.CLARK M7629534

Q.1 (a) Definitions

$\underline{D}$

The electric displacement vector:-

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$$

where $\underline{P}$ is the polarisation vector of a dielectric (dipole moment per unit volume of), $\underline{E}$ is the electric field vector and ε_0 is the permittivity of free space.

$\underline{H}$

The magnetic intensity vector :-

$$\underline{H} = \frac{\underline{B}}{\mu_0} - \underline{M}$$

where $\underline{B}$ is the magnetic field vector, μ_0 is the permeability of free space and $\underline{M}$ is the magnetisation vector (magnetic moment per unit volume) of a magnetic material.

$\int_{CS} \underline{P}.d\underline{S} = -q_{pol}$

$\oint \underline{M}.d\underline{l} = I_{mag}$

Both vectors are 'artificial' in the sense that they are merely defined from other physically observable vector quantities and do not impart any new information; they do, however, simplify problems of field calculations in certain situations.

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(b) Relation to Sources

$\underline{D}$ has its source only in free charges (q_{free}). This can be shown by using Gauss' flux theorum on the definition equation:-

$$\int_{CS} \underline{D}.d\underline{S} = \varepsilon_0 \int_{CS} \underline{E}.d\underline{S} + \int_{CS} \underline{P}.d\underline{S}$$

As the total electric field $\underline{E}$ results from all charges (q_{total}) and the polarisation $\underline{P}$ results only from polarisation charges (q_{polar}) this gives:-

$$\int_{CS} \underline{D}.d\underline{S} = q_{total} - q_{polar} = q_{free}$$

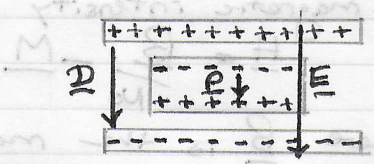
At first sight it would seem that $\underline{H}$ has its source only in free currents (i_{free}). This can also be shown by calculating the closed line integrals (or circulations) of the definition equation:-

$$\oint \underline{H}.d\underline{l} = \frac{1}{\mu_0}\oint \underline{B}.d\underline{l} - \oint \underline{M}.d\underline{l}$$

As the total magnetic field $\underline{B}$ results from all currents (i_{total}) and the magnetisation $\underline{M}$ results only from amperian currents (i_{mag}) (which account for the magnetic properties of matter) :-

$$\oint \underline{H}.d\underline{l} = i_{total} - i_{mag} = i_{free}$$

This is illustrated in the following diagram of a dielectric slab between the plates of a capacitor:-



The divergence theorum can be used to relate the surface integral of $\underset{\sim}{D}$ to the volume integral of its' divergence :-

$$\int_{cs} \underset{\sim}{D} \cdot d\underset{\sim}{S} = \int_{V} \operatorname{div} \underset{\sim}{D} \, dV = \int_{V} \rho_{free} \, dV$$

As this holds for any volume :-

$$\operatorname{div} \underset{\sim}{D} = \rho_{free}$$

Thus the divergence of $\underset{\sim}{D}$ equates to the free charge density, as a source

Good

$\frac{2}{3}$

Also express in differential form:

(4 marks) $\operatorname{div} \underline{D} = \rho_{free}$ (scalar eq)

However, the $\underset{\sim}{H}$ field is also affected by magnetic matter. Indeed, in a similar manner to dielectrics, the sources of $\underset{\sim}{H}$ are related to the surface integral :-

$$\int_{cs} \underset{\sim}{H} \cdot d\underset{\sim}{S}$$

The situation is not a direct analogy, however, as no "free magnetic charges" are known to exist!

In free space, or uniform, linear matter

$$\int_{cs} \underset{\sim}{H} \cdot d\underset{\sim}{S} = 0$$

However at the boundary of a magnetic material

$$\int_{cs} \underline{H} \cdot d\underline{S} = - \int_{cs} \underline{M} \cdot d\underline{S}$$

As suggested by a rearrangement of the definition equation :-

$$\underline{B} = \mu_0(\underline{H} + \underline{M})$$

The discontinuity of $\underline{M}$ at the boundary gives rise to sources of $\underline{H}$ (and sinks of $\underline{H}$)

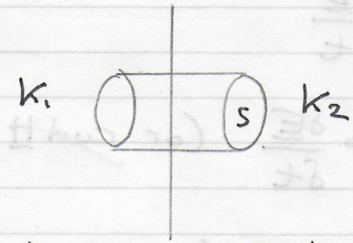
Furthering the analogy with the dielectric polarisation charge, these sources of $\underline{H}$ can be thought of as caused by a fictitious magnetic charge (q_{mag})

$$\int_{cs} \underline{H} \cdot d\underline{S} = \text{'}q_{mag}\text{'}$$

$\operatorname{curl} \underline{H} = \underline{J}_{free}$ (vector eq)

Q1 (cont)

(c) Boundary Conditions



Resulting from the fact that the net flux of $\underline{D}$ across the boundary of differing dielectrics is zero (as there are no charges enclosed in a Gaussian surface)

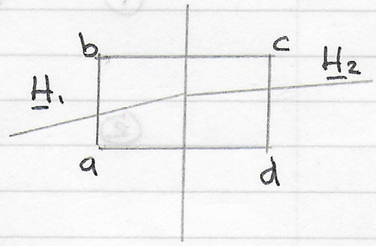
$$\int_{S_1} \underset{\sim}{D_1} \cdot d\underset{\sim}{S} + \int_{S_2} \underset{\sim}{D_2} \cdot d\underset{\sim}{S} = 0$$

$$\Rightarrow - \int_{S_1} D_{n_1} dS + \int_{S_2} D_{n_2} dS = 0$$

It is clear that the normal components are equal

$$D_{n_1} = D_{n_2}$$

5/5



As there are no real currents at the boundary of differing magnetic materials, the circulation is zero :-

$$\int_a^b \underset{\sim}{H_1} \cdot d\underset{\sim}{l_1} + \int_c^d \underset{\sim}{H_2} \cdot d\underset{\sim}{l_2} = 0$$

$$\Rightarrow - \int_a^b H_{t_1} dl_1 + \int_c^d H_{t_2} dl_2 = 0$$

Thus here the tangential components are equal

$$H_{t_1} = H_{t_2}$$

(d) Ease of Problems

Taking the example of a toroidal capacitor; it is apparant that lines of $\underline{D}$ are radial between the inner and outer radii Here,

4/4

$$\int_{CS} \underline{D} \cdot d\underline{S} = q_{free}$$

26/30

To find $\underline{D}$ at a given radius, a Gausian surface must be drawn to calculate the surface integral. this procedure is clearly complicated by the fact that the lines of $\underline{D}$ diverge between the two radii.

+5 (for clarity of expression)

The calculation of $\underline{H}$ for a toroidal solenoid is particularly simple because $\underline{B}$ and $\underline{H}$ are zero outside the coil and thus $\underline{H}$ depends only on the free current in the coil:-

$$H = \frac{Ni}{L}$$

4/4

This procedure is straight forward because no end effects need to be taken into account as sources of $\underline{H}$ and because $\underline{H}$ is always parallel to the axis of the coil.

Q 2/ (a) Maxwells' equations in differential form are:-

① $\text{curl } \underline{E} = -\frac{\delta \underline{B}}{\delta t}$

② $\text{curl } \underline{B} = \mu_0 \varepsilon_0 \frac{\delta \underline{E}}{\delta t}$ (or $\text{curl } \underline{H} = \underline{J}_f + \frac{\delta \underline{D}}{\delta t}$)

③ $\text{div } \underline{E} = 0$ (or $\text{div } \underline{D} = \rho_f$)

④ $\text{div } \underline{B} = 0$

Taking the curl of ②

$$\text{Curl}(\text{Curl } \underline{B}) = \nabla \times (\nabla \times \underline{B}) = \mu_0 \varepsilon_0 \frac{\delta}{\delta t}(\nabla \times \underline{E})$$

$$= \mu_0 \varepsilon_0 \frac{\delta}{\delta t}(\text{Curl } \underline{E})$$

$$= -\mu_0 \varepsilon_0 \frac{\delta^2 \underline{B}}{\delta t^2}$$

thus $\nabla \times (\nabla \times \underline{B}) = -\mu_0 \varepsilon_0 \frac{\delta^2 \underline{B}}{\delta t^2}$

From the given vector identity

$$\nabla(\nabla \times \underline{X}) = \nabla(\nabla \cdot \underline{X}) - \nabla^2 \underline{X}$$

Here $\nabla(\nabla \cdot \underline{B}) = \text{div } \underline{B} = 0$

$$\Rightarrow -\mu_0 \varepsilon_0 \frac{\delta^2 \underline{B}}{\delta t^2} = -\nabla^2 \underline{B}$$

$$\Rightarrow \nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\delta^2 \underline{B}}{\delta t^2}$$

Q2 (cont)

The complex wave

$$\underline{B} = B_0 \hat{y} \exp(j(\omega t - kz))$$

or $B_y = B_{y_0} e^{j(\omega t - kz)}$

must be shown to obey the wave equation.

Now $\nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\delta^2 \underline{B}}{\delta t^2}$

Can be interpreted as

$$\frac{\delta^2 B_y}{\delta z^2} = \mu_0 \varepsilon_0 \frac{\delta^2 B_y}{\delta t^2}$$

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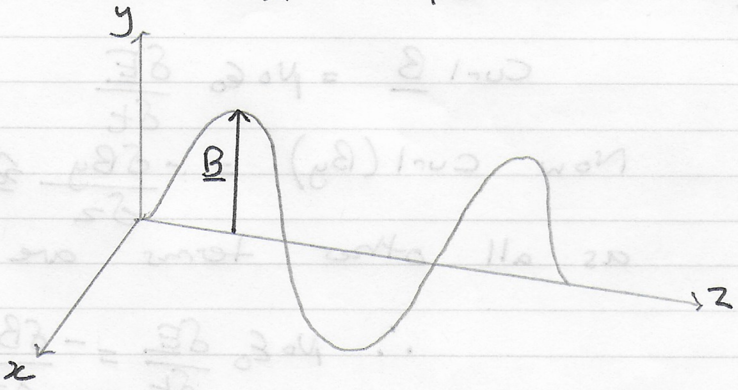
if the wave propagates in the z direction with the transverse $\underline{B}$ component in the y direction:-



Comparing the above equation to the standard one dimensional wave equation:-

$$\frac{\delta^2 F}{\delta x^2} = \frac{1}{V^2} \frac{\delta F}{\delta t^2}$$

it is clear that this is a wave of propagation velocity $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$

2/2

$$\therefore \frac{\delta^2 B_y}{\delta z^2} = \frac{1}{c^2} \frac{\delta^2 B_y}{\delta t^2}$$

Substituting $B_y = B_{y_0} e^{j(\omega t - kz)}$

into this wave equation, we have

$$-k^2 B_y = -\frac{1}{c^2}\omega^2 B_y$$

$$\Rightarrow \quad k^2 = \frac{\omega^2}{c^2}$$

$$\Rightarrow \quad k = \frac{\omega}{c}$$

This is both the relation between ω and k required and the proof that the complex wave satisfies the wave equations, as the above result is equivalent to $c = \lambda f$.

(b) from (2)

$$\text{Curl } \underline{B} = \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}$$

Now $\text{curl}(B_y) = -\frac{\delta B_y}{\delta z}\hat{x}$

as all other terms are zero.

$$\therefore \quad \mu_0 \varepsilon_0 \frac{\delta \underline{E}}{\delta t} = -\frac{\delta B_y}{\delta z}\hat{x}$$

If this equation holds for the given equation for $\underline{B}$ and for the corresponding equation for $\underline{E}$:-

$$\underline{B} = B_0 \hat{y} \exp(j(\omega t - kz))$$

$$\underline{E} = E_0 \hat{x} \exp(j(\omega t - kz))$$

Then the equation for $\underline{E}$ is valid.

Now $\mu_0 \varepsilon_0 \frac{\delta \underline{E}}{\delta t} = \mu_0 \varepsilon_0 j\omega E_0 \hat{x} \exp(j(\omega t - kz))$

$$= \mu_0 \varepsilon_0 \omega E_0 [\hat{x} \exp(j(\omega t - kz))$$

Q2 (cont.)

$$\text{and } -\frac{\delta B}{\delta t} = jkB_0\,\hat{x}\exp(j(\omega t - kz))$$

$$= kB_0[\hat{x}\exp(j(\omega t - kz))$$

Thus we must prove that:-

$$\mu_0\varepsilon_0\omega E_0 = kB_0$$

Now $K = \frac{\omega}{c}$ $\Rightarrow \mu_0\varepsilon_0\omega E_0 = \frac{\omega B_0}{c}$

and $\frac{1}{c} = \sqrt{\mu_0\varepsilon_0}$

$$\Rightarrow \mu_0\varepsilon_0 E_0 = \omega\sqrt{\mu_0\varepsilon_0}\,B_0$$

Finally the amplitudes of E_0 and B_0 are related by

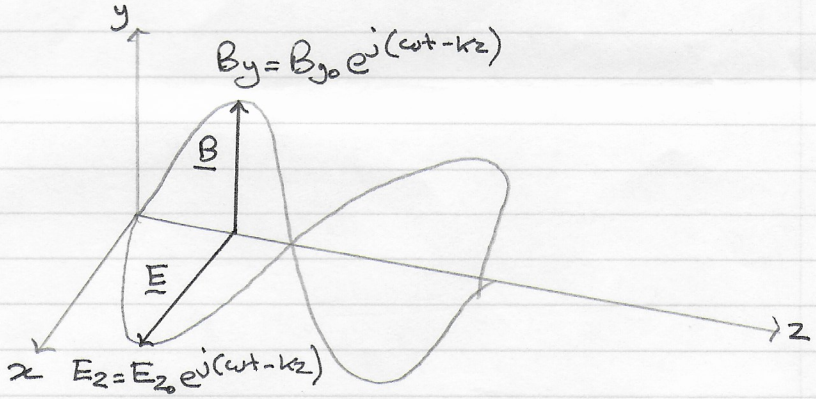
$$B = \frac{E}{c} = E\sqrt{\mu_0\varepsilon_0}$$

5/5

$$\Rightarrow \mu_0\varepsilon_0\omega E_0 = \omega\left(\sqrt{\mu_0\varepsilon_0}\right)^2 E_0$$

Thus the equation for $\underline{E}$ is uphold.

This is not surprising, as it follows from the orthogonal, complematary nature of the $\underline{E}$ and $\underline{B}$ components of an electromagnetic plane wave, propagating in the z direction:-



Now in free space $B = \mu_0 H$

Thus from $B = \frac{E}{c}$, $\mu_0 H = \frac{E}{c}$

$$\Rightarrow E = \mu_0 c H$$

The wave impedance of free space

$$Z_0 = \mu_0 c = \frac{E}{H} = \frac{\mu_0}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Dimensionally :-

$$Z_0 = \frac{E}{H} = \frac{[Vm^{-1}]}{[Am^{-1}]} = \frac{[V]}{[A]}$$

From Ohm's Law, $R = \frac{V}{i} = \frac{[V]}{[A]} = [\Omega]$

Thus Z_0 is measured in Ohms, Ω.

$$Z_0 = \mu_0 c = 1.26 \times 10^{-6} \times 299792458$$

$$\Rightarrow Z_0 = 376.73 \Omega \approx 377 \Omega$$

5/5

Q2 (cont).

(c)

$$\underline{E}' = \underline{E}\cos\theta + c\underline{B}\sin\theta \quad , \underline{B}' = -\frac{\underline{E}}{c}\sin\theta + \underline{B}\cos\theta$$

where $\theta = k$.

must be shown to obey Maxwell's equations.

① $$\text{Curl } \underline{E}' = -\frac{\delta \underline{B}'}{\delta t}$$

Left hand side:

Now $$\text{curl } \underline{E}' = \text{Curl}(\underline{E}\cos\theta + c\underline{B}\sin\theta)$$

$$\text{Curl } \underline{E} = -\frac{\delta B}{\delta t} \quad \text{and} \quad \text{curl } \underline{B} = \mu_0\varepsilon_0 \frac{\delta E}{\delta t}$$

$$\Rightarrow \text{Curl } \underline{E}' = -\frac{\delta B}{\delta t} -\sin\theta + \mu_0\varepsilon_0 c \frac{\delta E}{\delta t}\cos\theta$$

$$\text{as } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}, \quad \mu_0\varepsilon_0 c = \frac{\mu_0\varepsilon_0}{\sqrt{\mu_0\varepsilon_0}} = \sqrt{\mu_0\varepsilon_0} = \frac{1}{c}$$

$$\Rightarrow \text{Curl } \underline{E}' = \frac{\delta B}{\delta t}\sin\theta + \frac{1}{c}\frac{\delta E}{\delta t}\cos\theta \quad (a)$$

Right hand side:

$$-\frac{\delta B'}{\delta t} = -\frac{\delta}{dt}\left(-\frac{\underline{E}}{c}\sin\theta + \underline{B}\cos\theta\right)$$

$$= \frac{1}{c}\frac{\delta E}{\delta t}\cos\theta + \frac{\delta \underline{B}}{\delta t}\sin\theta \qquad (b)$$

As (a) and (b) are equal, equation ① is upheld.

$\frac{4}{4}$

(2) $\text{Curl } \underline{B}' = \mu_0 \epsilon_0 \frac{\delta \underline{E}'}{\delta t}$

Left hand side:

Now $\text{Curl } \underline{B}' = \text{Curl}\left(-\frac{\underline{E}}{c} \sin\theta + \underline{B} \cos\theta\right)$

$= \frac{1}{c} \frac{\delta \underline{B}}{\delta t} \cos\theta + {}^{-} \mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t} \sin\theta$ (c)

Right hand side:

$\mu_0 \epsilon_0 \frac{\delta \underline{E}'}{\delta t} = \mu_0 \epsilon_0 \frac{\delta}{\delta t}\left(\underline{E} \cos\theta + c\underline{B} \sin\theta\right)$

$\frac{3}{3}$ $\frac{\cancel{4}}{\cancel{4}}$ $= -\mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t} \sin\theta + \mu_0 \epsilon_0 c \frac{d\underline{B}}{\delta t} \cos\theta$

$= -\mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t} \sin\theta + \frac{1}{c} \frac{\delta \underline{B}}{\delta t} \cos\theta$ (d)

As (c) and (d) are equal, equation (2) is upheld.

(3) $\text{div } \underline{E}' = 0$

$\Rightarrow \text{div } \underline{E}' = \text{div}\left(\underline{E} \cos\theta + c\underline{B} \sin\theta\right) = 0$ $\frac{1}{1}$

As $\text{div } \underline{E} = \text{div } \underline{B} = 0$, this is upheld.

(4) $\text{div } \underline{B}' = 0$

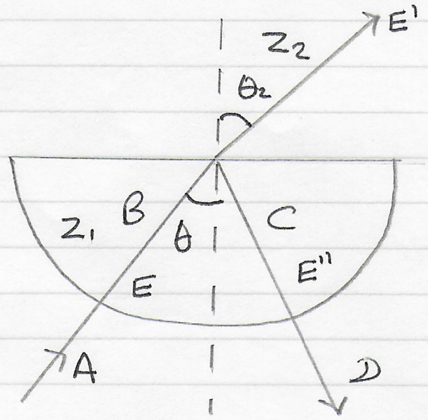
$\Rightarrow \text{div } \underline{B}' = \text{div}\left(-\frac{\underline{E}}{c} \sin\theta + \underline{B} \cos\theta\right)$ $\frac{1}{1}$

Again, as $\text{div } \underline{E} = \text{div } \underline{B} = 0$

this equation and thus all of Maxwell's equations are upheld!

$\frac{30}{30}$

Q3/



(a) The Brewster angle θ_B, is the angle at which full polarisation occurs:-

$$\tan \theta_B = \frac{n_2}{n_1}$$

As the first and last boundaries are crossed at the normal, the max polarisation occurs when the light ray is incident at the glass/air boundary, B/C.

$$\therefore \quad n_1 = 1.5 \ , n_2 = 1$$

$$\Rightarrow \tan \theta_B = \frac{1}{1.5}$$

$$\Rightarrow \underline{\theta_B = 33.69^\circ}$$

✓ $\frac{3}{3}$

Q3 (Cont) (b)

A/B Boundary $n_1 = 1$ $n_2 = 1.5$

As the incidence is normal,

$$E_6' = \frac{2n_1}{n_1+n_2} E_6$$

$$\Rightarrow E_6' = \frac{2\times 1}{1+1.5} E_6$$

$$\Rightarrow E_6' = 0.8 E_6$$

Thus $E_{6B} = 0.8 E_{6A}$ (i)

B/C Boundary.

Here some of the amplitude is lost to the refracted ray.

As $N_{glass} \approx N_{air}$, we can use :-

$$E_6'' = \frac{-\sin(\theta - \theta_2)}{\sin(\theta + \theta_2)} E_6$$

for the amplitude of the reflected ray.

$n_1 = 1.5$
$n_2 = 1$

Using Snell's law $n_1 \sin\theta = n_2 \sin\theta_2$

$$\Rightarrow \sin\theta_2 = \frac{n_1 \sin\theta}{n_2}$$

$$\Rightarrow \sin\theta_2 = 1.5 \times \sin 33.69°$$

$$\Rightarrow \theta_2 = 56.3°$$

$$\Rightarrow E_6'' = -\frac{\sin(33.69 - 56.3)}{\sin(33.69 + 56.3)} E_6$$

$$\Rightarrow E_6'' = 0.384 E_6$$

Thus $E_{6C} = 0.384 E_{6B}$ (ii)

C/D Boundary $n_1 = 1.5$ $n_2 = 1$

Here, again

$$E'_6 = \frac{2n_1}{n_1 + n_2} E_6$$

$$\Rightarrow E'_6 = \frac{2 \times 1.5}{1.5 + 1}$$

$$\Rightarrow E'_6 = 1.2\, E_6$$

Thus $E_{6D} = 1.2 E_{6C}$ (iii)

Combining (i), (ii) and (iii), we have

$$E_{6D} = 1.2 \times 0.384 \times 0.8\, E_{6A}$$

$$\Rightarrow \underline{E_{6D} = 0.368\, E_{6A}}$$

V. good

$\frac{13}{13}$

Q3(cont) (c)

The energy flux is given by the Poynting Vector;

$$\underset{\sim}{S} = \underset{\sim}{E} \times \underset{\sim}{H}$$

Now $S = \frac{1}{2}\frac{E^2}{Z}$

Thus for the calculations where the ray is travelling in glass, Z_{glass} will be required:-

A/B boundary $\quad E_A \rightarrow E_B \quad Z_0 \quad Z_{glass}$

B/C boundary $\quad E_B \rightarrow E_{\sigma C}$ polarised, $E_{\pi B}$ lost $\quad Z_{glass} \quad Z_{glass}$

C/D boundary $\quad E_{\sigma C} \rightarrow E_{\sigma D} \quad Z_{glass} \quad Z_0$

At the B/C boundary $\quad Z_1 = Z_{glass} \quad Z_2 = Z_0$

Now $$E_\sigma - E_\sigma'' = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_1} E_\sigma'$$

From (b) $E'' = 0.384 E_\sigma$

$$\Rightarrow E_\sigma - 0.384 E_\sigma = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_1} E_\sigma'$$

$$\Rightarrow 0.616 E_\sigma = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_1} E_\sigma'$$

Now $E_\sigma + E_\sigma'' = E_\sigma'$

$$\Rightarrow E_\sigma + 0.384 E_\sigma = E_\sigma'$$

$$\Rightarrow 1.384 E_\sigma = E_\sigma'$$

$$\therefore 0.616 E_\sigma = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_1} 1.384 E_\sigma$$

$$\Rightarrow \frac{0.616 E_6}{1.384 E_6} = \frac{Z_1 \cos 56.3^\circ}{377 \cos 33.69^\circ}$$

$$\Rightarrow 0.4458 = Z_1 \times 1.7686 \times 10^{-3}$$

$$\Rightarrow Z_1 = \frac{0.4458}{1.7686 \times 10^{-3}}$$

$$\Rightarrow Z_1 (Z_{glass}) = 252\ \Omega$$ 2/2

At A:

$$S_A = \frac{E_A^2}{2Z_0}$$

$$\Rightarrow S_A = \frac{E_A^2}{2 \times 377}$$

$$\Rightarrow S_A = 1.326 \times 10^{-3} E_A^2 \text{ wm}^{-2}$$ ✓ 2/2

At B:

$$S_B = \frac{E_B^2}{2Z_{glass}}$$

As the incidence is normal :-

$$E' = \frac{2n_1}{n_2 + n_1} E$$

Thus the relation found in (b) for E_6 holds for E

$$\therefore E_B = 0.8 E_A$$

$$\Rightarrow S_B = \frac{(0.8 E_A)^2}{2 \times 252}$$

$$\Rightarrow S_B = \frac{0.8^2 E_A^2}{504}$$

$$\Rightarrow S_B = 1.269 \times 10^{-3} E_A^2 \text{ wm}^{-2}$$ 2/2

Q3 (cont) At C:

$$S_C = \frac{E_c^2}{2 Z_{glass}}$$

At this boundary the E_π component is lost because of polarisation,

Now $|\underline{E}| = \sqrt{E_\pi^2 + E_\sigma^2}$

We can ignore the E_π component because we have worked throughout thus far with E_σ

Assuming the π and σ components were equal in magnitude

i.e. $E_\pi = E_\sigma$

$$E = \sqrt{E_\sigma^2 + E_\sigma^2} = \sqrt{2E_\sigma^2} = \sqrt{2}\, E_\sigma$$

When the E_π component is lost:-

$$E = \sqrt{E_\sigma^2} = E_\sigma$$

Thus the E amplitude is reduced by a factor of $\sqrt{2}$ X

From (b), we also know that the E_σ component is reduced by a factor of 0.384 at this boundary :-

ie $E_c = .384 \times .8 E_A$

$$\Rightarrow E_C = \frac{0.384 \times 0.8\, E_A}{\cancel{\sqrt{2}}}$$

$$\Rightarrow E_c = 0.217\, E_A$$

$$\therefore S_C = \frac{(0.217\, E_A)^2}{2 \times 252} \times 2$$

$$\Rightarrow S_C = \frac{0.217^2 E_A^2}{504}$$

$\frac{1}{2}$

$$\Rightarrow S_C = 9.343 \times 10^{-5} E_A^2 \ \mathrm{Wm^{-2}}$$

$1.885 \times 10^{-4}\ \mathrm{Wm^{-2}} \times E_A$

At D:

$$S_D = \frac{E_D^2}{2Z_0}$$

Again $E' = \frac{2n_1}{n_2+n_1} E$

Thus from (b) $E_D = 1.2 E_C$

$$\Rightarrow E_D = 1.2 \times 0.217 E_A$$

$$\Rightarrow E_D = 0.260 E_A$$

$$\therefore S_D = \frac{(0.260 E_A)^2}{2 \times 377}$$

$$\Rightarrow S_D = \frac{0.260^2 E_A^2}{754}$$ See above

$\frac{1}{2}$

$$\Rightarrow S_D = 8.96 \times 10^{-5} E_A^2 \ \mathrm{wm}^{-2}$$

$\times 2$ $1.810 \times 10^{-4} E_A \mathrm{Wm}^{-2}$

Although E is greater at D than at C, H will be smaller at D and greater at C by a corresponding amount in order to comply with the Laws of Conservation of energy. X

$0/2$

Although the energy flux at D is smaller than at C, the electric field at D is greater than at C because ϵ is smaller at D than at C

$\frac{24}{28}$

Q4/

$$\operatorname{div}(\underline{A} \times \underline{B}) = \underline{B} \cdot (\operatorname{curl} \underline{A}) - \underline{A} \cdot (\operatorname{curl} \underline{B})$$

Left hand side

$$\underline{A} \times \underline{B} = (A_y B_z - A_z B_y)\hat{\underline{x}} + (A_z B_x - A_x B_z)\hat{\underline{y}} + (A_x B_y - A_y B_x)\hat{\underline{z}}$$

$$\Rightarrow \operatorname{div}(\underline{A} \times \underline{B}) = \frac{\delta}{\delta x}(A_y B_z - A_z B_y) + \frac{\delta}{\delta y}(A_z B_x - A_x B_z) + \frac{\delta}{\delta z}(A_x B_y - A_y B_x)$$

$$= \left(\frac{\delta}{\delta x} A_y B_z - \frac{\delta}{\delta x} A_z B_y\right) + \left(\frac{\delta}{\delta y} A_z B_x - \frac{\delta}{\delta y} A_x B_z\right) + \left(\frac{\delta}{\delta z} A_x B_y - \frac{\delta}{\delta z} A_y B_x\right)$$

$$= \left(A_y \frac{\delta B_z}{\delta x} + B_z \frac{\delta A_y}{\delta x}\right) - \left(A_z \frac{\delta B_y}{\delta x} + B_y \frac{\delta A_z}{\delta x}\right) + \left(A_z \frac{\delta B_x}{\delta y} + B_x \frac{\delta A_z}{\delta y}\right) - \left(A_x \frac{\delta B_z}{\delta y} + B_z \frac{\delta A_x}{\delta y}\right) + \left(A_x \frac{\delta B_y}{\delta z} + B_y \frac{\delta A_x}{\delta z}\right) - \left(A_y \frac{\delta B_x}{\delta z} + B_x \frac{\delta A_y}{\delta z}\right)$$

Terms :

$$= \overset{(+1)}{A_y \frac{\delta B_z}{\delta x}} + \overset{(+2)}{B_z \frac{\delta A_y}{\delta x}} - \overset{(-3)}{A_z \frac{\delta B_y}{\delta x}} - \overset{(-4)}{B_y \frac{\delta A_z}{\delta x}}$$

$$+ \overset{(+5)}{A_z \frac{\delta B_x}{\delta y}} + \overset{(+6)}{B_x \frac{\delta A_z}{\delta y}} - \overset{(-7)}{A_x \frac{\delta B_z}{\delta y}} - \overset{(-8)}{B_z \frac{\delta A_x}{\delta y}}$$

$$+ \overset{(+9)}{A_x \frac{\delta B_y}{\delta z}} + \overset{(+10)}{B_y \frac{\delta A_x}{\delta z}} - \overset{(-11)}{A_y \frac{\delta B_x}{\delta z}} - \overset{(-12)}{B_x \frac{\delta A_y}{\delta x}}$$

Right hand side:

$$\text{Curl } \underline{A} = \left(\frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z}\right)\hat{\underline{x}}$$

$$+ \left(\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x}\right)\hat{\underline{y}}$$

$$+ \left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)\hat{\underline{z}}$$

$$\Rightarrow \underline{B} \cdot (\text{Curl } \underline{A})$$

$$= B_x\left(\frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z}\right)$$

$$+ B_y\left(\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x}\right)$$

$$+ B_z\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)$$

$$= \left(B_x\frac{\delta A_z}{\delta y} - B_x\frac{\delta A_y}{\delta z}\right)$$

$$+ \left(B_y\frac{\delta A_x}{\delta z} - B_y\frac{\delta A_z}{\delta x}\right) \quad (i)$$

$$+ \left(B_z\frac{\delta A_y}{\delta x} - B_z\frac{\delta A_x}{\delta y}\right)$$

$$\text{Curl } \underline{B} = \left(\frac{\delta B_z}{\delta y} - \frac{\delta B_y}{\delta z}\right)\hat{\underline{x}}$$

$$+ \left(\frac{\delta B_x}{\delta z} - \frac{\delta B_z}{\delta x}\right)\hat{\underline{y}}$$

$$+ \left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)\hat{\underline{z}}$$

Q4 (cont)

$$\Rightarrow \underline{A}\cdot(\text{curl}\,\underline{B}) = A_x\left(\frac{\delta B_z}{\delta y} - \frac{\delta B_y}{\delta z}\right) + A_y\left(\frac{\delta B_x}{\delta z} - \frac{\delta B_z}{\delta x}\right) + A_z\left(\frac{\delta B_y}{\delta x} - \frac{\delta B_x}{\delta y}\right)$$

$$= \left(A_x\frac{\delta B_z}{\delta y} - A_x\frac{\delta B_y}{\delta z}\right) + \left(A_y\frac{\delta B_x}{\delta z} - A_y\frac{\delta B_z}{\delta x}\right) + \left(A_z\frac{\delta B_y}{\delta x} - A_z\frac{\delta B_x}{\delta y}\right) \quad (ii)$$

$$\Rightarrow \underline{B}\cdot(\text{curl}\,\underline{A}) - \underline{A}\cdot(\text{curl}\,\underline{B}) = (i)-(ii)$$

$$= \left[\left(B_x\frac{\delta A_z}{\delta y} - B_x\frac{\delta A_y}{\delta z}\right) + \left(B_y\frac{\delta A_x}{\delta z} - B_y\frac{\delta A_z}{\delta x}\right) + \left(B_z\frac{\delta A_y}{\delta x} - B_z\frac{\delta A_x}{\delta y}\right)\right]$$

$$- \left[\left(A_x\frac{\delta B_z}{\delta y} - A_x\frac{\delta B_y}{\delta z}\right) + \left(A_y\frac{\delta B_x}{\delta z} - A_y\frac{\delta B_z}{\delta x}\right) + \left(A_z\frac{\delta B_y}{\delta x} - A_z\frac{\delta B_x}{\delta y}\right)\right]$$

terms:

$$= \overset{(+6)}{B_x\frac{\delta A_z}{\delta y}} - \overset{(-12)}{B_x\frac{\delta A_y}{\delta z}} + \overset{(+10)}{B_y\frac{\delta A_x}{\delta z}} - \overset{(-4)}{B_y\frac{\delta A_z}{\delta x}} + \overset{(+2)}{B_z\frac{\delta A_y}{\delta x}} - \overset{(-8)}{B_z\frac{\delta A_x}{\delta y}}$$

$$- \overset{(-7)}{A_x\frac{\delta B_z}{\delta y}} + \overset{(+8)}{A_x\frac{\delta B_y}{\delta z}} - \overset{(-11)}{A_y\frac{\delta B_x}{\delta z}} + \overset{(+1)}{A_y\frac{\delta B_z}{\delta x}} - \overset{(-3)}{A_z\frac{\delta B_y}{\delta x}} + \overset{(+5)}{A_z\frac{\delta B_x}{\delta y}}$$

Comparison of the terms (1 to 12) show the equality of the left and right sides of the expression.

Using this equation:-

where $\underline{A} = A\hat{z}$, $(A = k)$

$\underline{B} \cdot (\text{curl}\,\underline{A}) - \underline{A} \cdot (\text{curl}\,\underline{B})$ and thus $\text{div}(\underline{A} \times \underline{B})$

$$= -A_z \frac{\delta B_y}{\delta x} - B_y \frac{\delta A_z}{\delta x} + A_z \frac{\delta B_x}{\delta y} + B_x \frac{\delta A_z}{\delta y}$$

As $A_z = A$, and A is a constant

$$\frac{\delta A_z}{\delta x} = \frac{\delta A_z}{\delta y} = 0$$

$$\therefore \text{div}(\underline{A} \times \underline{B}) = A_z \frac{\delta B_x}{\delta y} - A_z \frac{\delta B_y}{\delta x}$$

$$\Rightarrow \text{div}(\underline{A} \times \underline{B}) = A\left(\frac{\delta B_x}{\delta y} - \frac{\delta B_y}{\delta x}\right)$$

$\frac{12}{12}$